Tracial approximation and its application in crossed producted C^*-algebras

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Let A be a C^*-algebra.

- Denote by A_+ the set of positive elements in A,
- Denote by A_{sa} the set of self-adjoint elements in A,
- Denote by $\mathrm{P}(A)$ the set of projections in A,
- Denote by $\mathrm{PI}(A)$ the set of partial isometries in A,
- Denote by $\mathrm{GL}(A)$ the set of invertible elements in A,
- Denote by $\mathrm{U}(A)$ the set of unitary elements in A and $\mathrm{U}(A)_0$ the set of path connected component of 1_A of $\mathrm{U}(A)$.

- For a C^*-subalgebra C of A, we write $a \in_\epsilon C$ if there is $b \in C$ such that $\|a - b\| < \epsilon$.
- For $a, b \in A$, we write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$.

A unital C^*-algebra A is said to have stable rank one, and written as $\mathrm{tsr}(A) = 1$, if $\overline{\mathrm{GL}(A)} = A$.
A unital C^*-algebra A is said to have real rank zero, and written as $\mathrm{RR}(A) = 0$, if $\overline{\mathrm{GL}(A) \cap A_{sa}} = A_{sa}$.

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Let $0 < \sigma_2 < \sigma_1 < 1$. Define $f_{\sigma_2}^{\sigma_1}$ by

$$f_{\sigma_2}^{\sigma_1}(t) = \begin{cases} 1, & t \geq \sigma_1 \\ \text{linear}, & \sigma_2 < t < \sigma_1 \\ 0, & 0 \leq t \leq \sigma_2. \end{cases}$$

Let a and b be two positive elements in a C^*-algebra A. We write $[a] \leq [b]$, if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in \mathrm{Her}(a), v^*c, cv \in A, vv^* = P_a$, where P_a is the range projection of a in A^{**}, and $v^*cv \in \mathrm{Her}(b)$. We write $[a] = [b]$ if $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$. We write $n[a] \leq [b]$, if there exist orthogonal positive elements $b_1, \cdots, b_n$ in $\overline{bAb}$ such that $[a] \leq [b_i]$ for all i.

Denote by $\mathcal{I}^{(k)}$ the class of all the C^*-algebras which are unital hereditary C^*-subalgebras of C^*-algebras of the form $C(X) \bigotimes F$, where X is a k-dimensional finite CW complex and F is a finite dimensional C^*-algebra.

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The following definition was first given by H. Lin as a noncommutative analog of the covering dimension for a topological space X, which has close connection with classification of nuclear C^*-algebras.

Definition (H. Lin):

Let A be a unital C^*-algebra. A is said to have tracial rank no more than k (denoted by $\mathrm{TR}(A) \le k$) if for any $\epsilon > 0$, any finite subset F of A containing a nonzero positive element a, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, and any integer $n > 0$, there exist a projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{I}^{(k)}$ of A with $1_C = p$ such that

- $\|xp - px\| < \epsilon$ for all $x \in F$;
- $pxp \in_\epsilon C$ for all $x \in F$;
- $n[f_{\sigma_2}^{\sigma_1}(((1-p)a(1-p))] \le [f_{\sigma_4}^{\sigma_3}(pap)]$.

If $\mathrm{TR}(A) \le k$ but $\mathrm{TR}(A) \not\le k-1$, we write $\mathrm{TR}(A) = k$.

If A is a simple C^*-algebra, the above definition can be greatly simplified. For example, the above third condition can be replaced by $[1-p] \le [a]$.

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Following Lin's tracial approximation definition above, G. A. Elliott and Z. Niu considered unital simple C^*-algebras tracially approximated by certain C^*-algebras.

Definition

Let $\mathfrak{C}$ be a class of unital C^*-algebras. Then the class of C^*-algebras which can be tracially approximated by C^*-algebra in $\mathfrak{C}$, denoted by $\mathrm{TA}\mathfrak{C}$, is defined as follows. A simple unital C^*-algebra A is said to belong to the class $\mathrm{TA}\mathfrak{C}$, if for any $\epsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element $a \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and $B \in \mathfrak{C}$, such that (1) $\|xp - px\| < \epsilon$ for all $x \in F$, (2) $pxp \in_\epsilon B$ for all $x \in F$, (3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

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If a class $\mathfrak{C}$ is closed under finite direct sum, closed under tensor with matrix algebras or closed under taking unital hereditary C^*-subalgebras, then the class TA$\mathfrak{C}$ is also closed under passing to finite direct sum, matrix algebras or unital hereditary C^*-subalgebras.

Following Lin's tracial approximation for non-simple C^*-algebras, X. Fang and Q. Fan considered unital C^*-algebras tracially approximated by certain C^*-algebras.

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If A is a unital simple C^*-algebra, we have the following.

Theorem

Let $\mathfrak{C}$ be a class of unital C^*-algebras such that $\mathfrak{C}$ is closed under taking unital hereditary C^*-subalgebras and closed taking finite direct sums. Let A be a simple unital C^*-algebra. Then the definition of G. A. Elliott and Z. Niu. and the definition of X. Fang and Q. Fan. are equivalent.

The question of the behavior of C^*-algebra properties under passage from a class $\mathfrak{C}$ to the class TA$\mathfrak{C}$ is interesting and sometimes important. In fact the property of having tracial states, the property of being of stable rank one, and the property that the strict order on projections is determined by traces were used in the proof of the classification theorem by G. A. Elliott and Z. Niu.

The following properties of C^*-algebras in a class $\mathfrak{C}$ are inherited by simple C^*-algebras in the class TA$\mathfrak{C}$

- being stably finite ;
- having real rank zero ;
- having stable rank one;
- having the Blackadar comparison property;
- any state of the order-unit K_0-group comes from a tracial state of the algebra;
- the canonical map from the unitary group modulo the connected component containing the identity to the K_1-group being injective ;
- K_1-surjective property.

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Let A be a C^*-algebra, and let $M_n(A)$ denote the $n \times n$ matrices whose entries are elements of A. Let $M_\infty(A)$ denote the algebraic limit of the direct system $(M_n(A), \phi_n)$ where $\phi_n : M_n(A) \to M_{n+1}(A)$ is given by $a \mapsto diag(a, 0)$. Let $M_\infty(A)_+$(resp. $M_n(A)_+$) denote the positive elements in $M_\infty(A)$ (resp. $M_n(A)$). For positive element a and b in $M_\infty(A)$, write $a \oplus b$ to denote the element $diag(a, b)$, which is also positive in $M_\infty(A)$. Given $a, b \in M_\infty(A)_+$, we say that a is Cuntz subequivalent to b (written $a \lesssim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of element of $M_\infty(A)$ such that

$$\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0.$$

We say that a and b are Cuntz equivalent (Written $a \sim b$) if $a \lesssim b$ and $b \lesssim a$. We write $\langle a \rangle$ for the equivalent class of a.

The object

$$W(A) := M_\infty(A)_+/ \sim$$

will be called the Cuntz semigroup of A. Observe that $W(A)$ becomes a positively ordered abelian monoid when equipped with the operation

$$\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle$$

and the partial order

$$\langle a\rangle \leq \langle b\rangle \Leftrightarrow a \precsim b.$$

We define

$$Cu(A) = (A \otimes \mathbb{K})_+/ \sim .$$

We also call $Cu(A)$ the Cuntz semigroup of A.

We say that $W(A)$ has the cancelation property, if for any a, b, c be positive elements (all in $M_\infty(A)_+$), $a \oplus c \lesssim b \oplus c$ imply that $a \lesssim b$.

We say that $W(A)$ has the weak cancelation property, if for any a, b be positive elements, $a \oplus c \lesssim b \oplus (c - \epsilon)$ imply that $a \lesssim b$ for some positive element c and positive real number ϵ. Clearly if $W(A)$ has weakly cancelation property, then $a \oplus p \lesssim b \oplus p$ imply that $a \lesssim b$ for any nonzero projection p in $M_\infty(A)_+$), which is called projectional cancelation property.

The Cuntz semigroup is constructed using positive elements of a C^*-algebra in much the same way that Murray-von Neuman semigroup is constructed using projections.

For the possible lack of much projections in a C^*-algebra, the Cuntz semigroup has more capability to detect significantly more structure than Murray-von Neuman semigroup.

This sensitivity of the Cuntz semigroup makes it an excellent candidate to distinguish C*-algebras, both simple and non-simple. In the simple case, well-behaved algebras, it contains the same information as the Elliott invariant.

Toms used the Cuntz semigroup to distinguish simple, nuclear C^*-algebras which cannot be distinguished by the Elliott invariant.

A. Ciuperca and G. A. Elliott show that in the non-simple case, the Cuntz semigroup is a classifying invariant for all AI C^*-algebras (their approach relies on Thomsen’ s classification of AI C*-algebras).

It then became clear that a further study of the Cuntz semigroup is needed, in fact, it is very important in Elliott’s classification program.

M. Rordam, W. Winter show that $W(A)$ has the weak cancellation for any stable rank one C^*-algebra.

Theorem

Let A be a C^*-algebra of stable rank one, let a, b be positive elements in $M_\infty(A)$, and let p be a projection in $M_\infty(A)$ such that

$$a \oplus p \lesssim b \oplus p$$

Then $a \lesssim b$.

Recently, we show that the following theorem about Cuntz semigroup.

Theorem

Let $\mathfrak{C}$ be a class of unital C^*-algebras such that for any $B \in \mathfrak{C}$, the Cuntz semigroup $W(B)$ has the weak (or projectional) cancelation property. Then the Cuntz semigroup $W(A)$ has the weak (or projectional) cancelation property for any simple unital C^*-algebra $A \in TA\mathfrak{C}$.

Proof We show that for any $\varepsilon > 0$, $(a - 3\varepsilon)_+ \lesssim b$.

Without loss of generality, we may assume that there exist $d \in A$ such that $(a + p) = d(b + p)d^*$.

For $F = \{a,\ b,\ p,\ d,\ d^*\}$, any $\varepsilon > 0$, any $\delta > 0$, since $A \in TA\mathfrak{C}$, there exist a C^*-subalgebra A_n of A and a nonzero projection $q_n \in A$ with $A_n \in \mathfrak{C}$ and $1_{A_n} = q_n$, such that

$$\|a - a_n - \bar{a}_n\| < \varepsilon,$$

and

$$\|a_n + p_n + \bar{a}_n + \bar{p}_n - (\bar{d}_n + d_n)(b_n + p_n + \bar{b}_n + \bar{p}_n)({d_n}^* + \bar{d}_n^*)\| < \delta.$$

Therefore we have

$$\|a - (a_n - \varepsilon)_+ - (\bar{a}_n - \varepsilon)_+\| < 3\varepsilon,$$

$$\|a_n + p_n - d_n(b_n + p_n){d_n}^*\| < \delta,$$

$$\|\bar{a}_n + \bar{p}_n - \bar{d}_n(\bar{b}_n + \bar{p}_n)\bar{d}_n^*\| < \delta.$$

We have

$$(a_n + p_n - \delta)_+ \lesssim b_n + p_n,$$

$$(\bar{a}_n + \bar{p}_n - \delta)_+ \lesssim \bar{b}_n + \bar{p}_n.$$

Without loss of generality, we may assume that

$$(\bar{a}_n + \bar{p}_n - \delta)_+ = \bar{e}(\bar{b}_n + \bar{p}_n)\bar{e}.$$

Since $a_n,\ p_n,\ b_n,\ (a_n - \varepsilon)_+ \in A_n$, and $A_n \in \mathfrak{C}$(We chose sufficiently small δ), we have

$$(a_n - \varepsilon)_+ \lesssim b_n.$$

For $F=\{\bar{a}_n,\ \bar{p}_n,\ (\bar{a}_n-\varepsilon)_+,\ \bar{b}_n,\ (a_n-\varepsilon)_+\}\subseteq A$, any $\bar{\delta}>0$, Since $A\in TA\mathfrak{C}$. There exist a C^*-subalgebra B_m of A and a nonzero projection $r_m\in A$ with $B_m\in\mathfrak{C}$ and $1_{B_m}=r_m$, there exist $\bar{a}_{n,m},\ \bar{b}_{n,m},\ \bar{p}_{n,m}\in B_m$, and $\bar{\bar{a}}_{n,m},\ \bar{\bar{b}}_{n,m},\ \bar{\bar{p}}_{n,m}\in(1-r_m)A(1-r_m)$, such that

$$\|\bar{a}_n-\bar{a}_{n,m}-\bar{\bar{a}}_{n,m}\|<\varepsilon/3.$$

$$\begin{aligned}&\|(\bar{a}_{n,m}-\delta)_+ +\bar{p}_{n,m}+(\bar{\bar{a}}_{n,m}-\delta)_+ +\bar{\bar{p}}_{n,m}\\&-(\bar{e}_{n,m}+\bar{\bar{e}}_{n,m})(\bar{b}_{n,m}+\bar{\bar{b}}_{n,m}+\bar{p}_{n,m}+\bar{\bar{p}}_{n,m})(\bar{e}_{n,m}+\bar{\bar{e}}_{n,m})\|<\bar{\delta}\end{aligned}$$

Therefore We have

$$\|(\bar{a}_{n,m}-\delta)_+ +\bar{p}_{n,m}-\bar{e}_{n,m}(\bar{b}_{n,m}+\bar{p}_{n,m})\bar{e}_{n,m}\|<\bar{\delta},$$

$$\|(\bar{\bar{a}}_{n,m}-\delta)_+ +\bar{\bar{p}}_{n,m}-\bar{\bar{e}}_{n,m}(\bar{\bar{b}}_{n,m}+\bar{\bar{p}}_{n,m})\bar{\bar{e}}_{n,m}\|<\bar{\delta}.$$

We have

$$(\bar{a}_{n,m} - \delta - \bar{\delta})_+ + \bar{p}_{n,m} \lesssim \bar{b}_{n,m} + \bar{p}_{n,m},$$

and

$$(\bar{\bar{a}}_{n,m} - \delta - \bar{\delta})_+ + \bar{\bar{p}}_{n,m} \lesssim \bar{\bar{b}}_{n,m} + \bar{\bar{p}}_{n,m}.$$

Since $B_m \in \mathfrak{C}$, we have $(\bar{a}_{n,m} - \delta - \bar{\delta})_+ \lesssim \bar{b}_{n,m}$.

Since $[\bar{\bar{p}}_{n,m}] \leq [1 - r_m] \leq [(a_n - \varepsilon)_+]$, there exist $c \in A$ such that $[\bar{\bar{p}}_{n,m}] + [c] = [(a_n - \varepsilon)_+]$. We have ((We chose sufficiently small $\delta, \bar{\delta}$)

$$\begin{aligned}(a - 3\varepsilon)_+ &\lesssim (a_n - \varepsilon)_+ + (\bar{a}_n - \varepsilon)_+ \\ &\lesssim [\bar{\bar{p}}_{n,m}] + [c] + (\bar{a}_{n,m} - \varepsilon/3)_+ + (\bar{\bar{a}}_{n,m} - \varepsilon/3)_+ \\ &\lesssim [\bar{\bar{p}}_{n,m}] + [c] + (\bar{a}_{n,m} - \varepsilon/3)_+ + (\bar{\bar{b}}_{n,m} - \varepsilon/3)_+ \\ &\lesssim b_n + \bar{b}_{n,m} + \bar{\bar{b}}_{n,m} \lesssim b\end{aligned}$$

Therefore we have $a \lesssim b$.

The Rokhlin property in ergodic theory was adapted to the context of von Neumann algebras by A. Connes . It was adapted by R. Hermann and A. Ocneanu for UHF-algebras. M. Rordam and A. Kishimoto introduced the Rokhlin property in a more general C^*-algebra context. More recently, N. C. Phillips and H. Osaka studied finite group actions which have a certain type of Rokhlin property on simple C^*-algebras.

N. C. Phillips raised the question how to introduce an appropriate Rokhlin property for non-simple C^*-algebras.

For simple or non-simple C^*-algebras, we have the following definition of Rokhlin property of finite group actions.

Definition

Let A be an infinite dimensional finite separable unital C^*-algebra and $\alpha, G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the second tracial Rokhlin property if for every $\varepsilon > 0$, every finite set $\mathcal{F} \subseteq A$, every positive elements $b, x \in A$, there exist $g_0, g_1 \in G$ and mutually orthogonal projections $e_g \in A$ for $g \in G$ such that

- $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in \mathcal{F}$.
- $\|e_{g_0} x e_{g_0}\| > \|x\| - \varepsilon$.
- With $e = \sum_{g \in G} e_g$, $[\alpha_{g_1}(1 - e)] \leq [b]$.

Remark: This definition is first given by J. Hua and H. Lin for automorphism in $\mathbb{Z}$, and then used to finite group action by X. Yang and X. Fang.

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Example (X. Yang and X. Fang):

Suppose that G is a finite group and D is an infinite dimensional simple unital AF-algebra. Let $\beta : G \to \mathrm{Aut}(D)$ be an action satisfying the tracial Rokhlin property. Suppose that $\mathrm{card}(G) = n$, and we may write $G = \{g_1, g_2, \cdots, g_n\}$. Let $A = \overbrace{D \oplus D \oplus \cdots \oplus D}^{n}$, where $n > 1$, and define $\alpha : G \to \mathrm{Aut}(A)$ by

$$\alpha_{g_i}(a_1, a_2, \cdots, a_n)$$
$$= (\beta_{g_i}(a_i), \beta_{g_i}(a_2), \cdots, \beta_{g_i}(a_{i-1}), \beta_{g_i}(a_1), \beta_{g_i}(a_{i+1}), \cdots, \beta_{g_i}(a_n)).$$

It is obvious that A is non-simple but α-simple. Moreover, it is easy to check that α has the second tracial Rokhlin property.

If the original algebra is non-simple AF-algebra or A$\mathbb{T}$-algebra, we get the following results:

Theorem

Let A be an infinite dimensional unital AF-algebra and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Suppose that A is α-simple, then $\mathrm{TR}(A \times_\alpha G) = 0$.

Theorem

Let A be an infinite dimensional unital A$\mathbb{T}$-algebra with the (SP)-property and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Suppose that A is α-simple, then $TR(A \times_\alpha G) \leq 1$.

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Using the following theorem

Theorem

Let $\mathfrak{C}$ be a class of unital C^*-algebras. Then any simple unital C^*-algebra $A \in TA(TA\mathfrak{C})$ is a $TA\mathfrak{C}$ algebra.

we can prove the following theorem.

Theorem

Let $\mathfrak{C}$ be a class of unital C^*-algebras such that $\mathfrak{C}$ is closed under taking unital hereditary C^*-algebra and closed taking finite direct sums. Suppose that $B/I \in \mathfrak{C}$ for any $B \in \mathfrak{C}$ and every closed ideal I of B. Let $A \in TA\mathfrak{C}$ be an infinite dimensional α-simple unital C^*-algebra with the SP property. Suppose that $\alpha : G \to Aut(A)$ is an action of a finite group G on A which has the second tracial Rokhlin property. Then the crossed product algebra $C^*(G, A, \alpha) \in TA\mathfrak{C}$.

We have the following two corollary:

Corollary

Let A be an infinite dimensional unital α-simple tracial rank zero C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Then $\mathrm{TR}(C^*(G, A, \alpha)) = 0$.

Corollary

Let A be an infinite dimensional α-simple unital tracial topological rank one C^*-algebra with the SP property. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Then $\mathrm{TR}(C^*(G, A, \alpha)) = 1$.

We have the following two corollary:

Corollary

Let A be an infinite dimensional unital α-simple tracial rank zero C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Then $\mathrm{TR}(C^*(G, A, \alpha)) = 0$.

Corollary

Let A be an infinite dimensional α-simple unital tracial topological rank one C^*-algebra with the SP property. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the second tracial Rokhlin property. Then $\mathrm{TR}(C^*(G, A, \alpha)) = 1$.

We will prove the following theorem.

Therorem

Let $\mathfrak{C}$ be a class of unital C^*-algebras. Then any simple unital C^*-algebra $A \in TA(TA\mathfrak{C})$ is a $TA\mathfrak{C}$ algebra.

Proof We need to show that for any finite subset $F = \{b_1, b_2, \cdots, b_n\} \subseteq A$, any nonzero positive element a, and any $\varepsilon > 0$, there exist a projection $r \in A$ and a C^*-subalgebra D of A with $1_D = r$ and with $D \in \mathfrak{C}$, such that

(1) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(2) $rxr \in_\varepsilon D$ for all $x \in F$,

(3) $[1_A - r] \leq [a]$.

Since $A \in TA(TA\mathfrak{C})$, for any $\delta > 0$ and any finite subset $G \subseteq A$, there exist a projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and with B a member of $TA\mathfrak{C}$, such that

$(1)'$ $\|xp - px\| < \delta$ for all $x \in G$,

$(2)'$ $pxp \in_\delta B$ for all $x \in G$,

$(3)'$ $2[1_A - p] \leq [a]$.

If $[1_A - p] = 0$, we have $1_A = p$, A is a $TA\mathfrak{C}$ algebra.

Suppose that $[1_A - p] \neq 0$. By $(3)'$, there exist partial isometries $v_1,\ v_2 \in A$ such that $v_1^* v_1 = 1_A - p,\ v_2^* v_2 = 1_A - p,\ v_1 v_1^*,\ v_2 v_2^* \in Her(a)$ and $(v_1 v_1^*)(v_2 v_2^*) = 0$. Set $a_1 = v_1 v_1^*,\ a_2 = v_2 v_2^*$. Then we have $a_1 \neq 0,\ a_2 \neq 0,\ a_1,\ a_2 \in Her(a),\ a_1 a_2 = 0$.

Since A is a simple unital C^*-algebra, there are $x_i \in A (i = 1, 2, \cdots, k)$ such that $\Sigma_{i=1}^{k} x_i a_1 x_i^* = 1$.

Since $A \in TA(TA\mathfrak{C})$, for any $\varepsilon > 0$, there exist a projection $t \in A$ and a C^*-subalgebra C of A with $1_C = t$ and $C \in TA\mathfrak{C}$, such that

$(1)''$ $\|xt - tx\| < \varepsilon$ for all $x \in H$,

$(2)''$ $txt \in_\varepsilon C$ for all $x \in H$, and $\|ta_1 t\| \geq \|a_1\| - \varepsilon$,

$(3)''$ $[1_A - t] \leq [a_2]$, where
$H = F \cup \{a_1, x_1, x_2, \cdots, x_k, x_1^*, x_2^*, \cdots, x_k^*\}$.

By $(1)''$ and $(2)''$ there exist $a_1' \in C$ and $a_1'' \in (1_A - t)A(1_A - t)$ such that $\|a_1 - a_1' - a_1''\| < 2\varepsilon$. We have $a_1' \neq 0$ and $[a_1'] \leq [a_1]$.

By $(1)''$ and $(2)''$ there exist x_1', x_2' $\cdots$, x_k' in C, such that $\Sigma_{i=1}^k x_i' a_1' x_i'^* = t$. Take $0 < d_1 < d_2 < 1$ such that $\|\Sigma_{i=1}^k x_i' {a_1'}^{1/2} f_{d_1}^{d_2}(a_1') {a_1'}^{1/2} x_i'^* - t\| < 1$. Put $z = (\Sigma_{i=1}^k x_i' a_1'^{1/2} f_{d_1}^{d_2}(a_1') a^{1/2} x_i'^*)^{-1}$, $y_i = z^{-1/2} x_i' a_1'^{1/2}$. Then we have $\Sigma_{i=1}^k y_i f_{d_1}^{d_2}(a_1') y_i^* = t$.

For any $\varepsilon > 0$, any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, we may assume that $\sigma_4 < d_3 < d_4 < d_1 < d_2 < \sigma_1$ and $\|a\| \leq 1$, since $C \in TA\mathfrak{C}$, there exist a projection $r \in A$ and a C^*-subalgebra D of C with $1_D = r$ and $D \in \mathfrak{C}$, such that

$(1)'''$ $\|rx - xr\| < \varepsilon$ for all $x \in H'$,

$(2)'''$ $rxr \in_\varepsilon D$ for all $x \in H'$,

$(3)'''$ $k[f_{d_1}^{d_2}((t-r)a_1'(t-r))] \leq [f_{d_3}^{d_4}(ra_1'r)]$, where $H' = \{tb_1t,\ tb_2t,\ \cdots,\ tb_nt,\ a_1', x_1', x_2',\ \cdots,\ x_k',\ y_1,\ y_2',\ \cdots,\ y_k,\ y_1^*,\ y_2'^*,\ \cdots,\ y_k^*,\ f_{d_1}^{d_2}(a_1'),\ t\}$, By functional calculus, we have

$\|\Sigma_{i=1}^k (t-r)y_i(t-r)f_{d_1}^{d_2}((t-r)a_1'(t-r))(t-r)y_i^*(t-r)-(t-r)\| < 1.$

So there are $z_i \in (t-r)C(t-r)$ such that

$$\Sigma_{i=1}^k z_i f_{d_1}^{d_2}((t-r)a_1'(t-r))z_i^* = t-r.$$

We have

$$[t-r] \leq k[f_{d_1}^{d_2}((t-r)a(t-r))].$$

Therefore,

$$\begin{aligned}[t-r] &\leq k[f_{d_1}^{d_2}((t-r)a_1'(t-r))] \leq [f_{d_3}^{d_4}(ra_1'r)] + [f_{d_3}^{d_4}((t-r)a_1'(t-r))] \\ &\leq [f_{\sigma_3}^{\sigma_4}(a_1')] \leq [a_1'].\end{aligned}$$

We have

(1) $\|xr - rx\| < 3\varepsilon$ for all $x \in F$,

(2) $rxr \in_{3\varepsilon} D$ for all $x \in F$,

(3) $[1_A - r] \leq [1_A - t] + [t-r] \leq [a_2] + [a_1'] \leq [a_2] + [a_1] \leq [a]$.

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Thanks!